Minimal Shape-preserving Projections Onto Π_n: Generalizations and Extensions.

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Abstract

The goal of this paper is to further the investigation begun in [6]. With the benefit of nearly ten years of work, we begin by indicating how several proofs from [6] can be substantially improved. We show that the problem of preserving k-convexity onto Π_n is one part of a larger shape-preserving problem (*multi-convex* preservation) relative to Π_n and we completely solve this expanded problem. And finally we demonstrate that multi-convex preserving projections constructed in this paper are in fact of minimal operator norm in a large class of Banach spaces.

1 Introduction

The general problem of approximating elements of Banach space X from a fixed subspace V has a long history, with a variety of special-case considerations which are considered classical. The determination of a best approximation is an example of such a problem and occupies a large position in the literature on mathematical analysis. Related to the problem of best-approximation is the *minimal projection* question. A *projecton* $P : X \to V$ is a linear operator with the property that P resticted to V is the identity operator (i.e., $Pv = v$ for all $v \in V$). We denote by $\mathcal{P}(X, V) = \mathcal{P}$ the set of all projections from X onto V. Whenever $\mathcal{P} \neq \emptyset$ we may look for an element $P_0 \in \mathcal{P}$ such that $\|P_0\| \leq \|P\|$ for all $P \in \mathcal{P}$. We say that such a projection P_0 is a *minimal projection*. It is worth noting that there exists a large number of papers concerning minimal projections (a variety of specific considerations is represented in [1], [2], [3], [8] and [9]).

From the viewpoint of approximation, it is of added benefit when there exist elements of $\mathcal{P}$ that leave invariant particular characteristics of elements from X. For example, if X is the Banach space $C^2(K)$ (space of real-valued functions with continuous second derivative, normed via $\max_{i=0,1,2}\{\|f^i\|_\infty\}$ and $V = \Pi_3$ (space of third degree algebraic polynomials), we might ask for projections $P \in \mathcal{P}$ which maps positive functions to positive cubics or perhaps monotone functions onto onto cubics. Unfortunately, neither of these requests can be fulfilled - there does not exist $P \in \mathcal{P}$ preserving either positivity or monotonicity. There does exist however a convexity-preserving projection from X onto V. Thus we are led to a natural question: given X, V and $S \subset X$, does there exist $P \in \mathcal{P}$ such that $PS \subset S$? We refer to such a P as a *shape-preserving projection*. While in general this question is unsolved, there are satisfactory results in particular settings (see for example [5], [13], [15] and [7]).

The paper [6] was the first attempt (as far as we can tell) to combine minimal projection theory and existence of shape-preserving projections. The context of that paper is straightforward: fix positive integer n and choose integer $r \geq n$. Let $X = (C^r[0,1], \|\cdot\|)$ and $V = \Pi_n$, where $\|f\| = \max_{i=0..r}\{\|f^{(i)}\|_\infty\}$. For integer $j \in [0,n]$, let $S_j \subset X$ denote the set of j-convex functions, where $f \in X$ is said to be *j-convex* if $f^{(j)}$ is nonnegative on $[0,1]$. The main results of [6] show that there exists $P \in \mathcal{P}$ preserving j-convexity if and only if $j \geq n-1$. In the case $j = n-1$, the norm of a minimal projection is $3/2$ for all n.

There are a variety of way to extend these results; we mention here some of work in progress towards these extensions. In the paper [14], $C^r[0,1]$ is replaced by $C[0,1]$ and j-convexity is defined via divided-differences. In that setting, the results are quite 'negative', in the sense that, for example, there does not exist a mononticity-preserving projection from $C[0,1]$ onto Π_2 (note the contrast with the result from [6]). Roughly speaking, in order to preserve $(n-1)$-convexity onto n-dimensional subpaces of $C[0,1]$, one cannot employ polynomial spaces. In the paper [11], Π_n is replaced by more general subspaces of $C^r[0,1]$ and a *multi-convex* shape is preserved (*multi-convexity* will be discussed in detail below). In the paper [12], $C^r[0,1]$ is replaced by multivariable functions defined on rather general compact subsets of $\mathbb{R}^k$. The subspaces and shapes considered in that context are formed via tensor products.

In this paper we generalize [6] in the directions of shape and norm, as well provide corrected and shorten proofs of two results therein. Specifically, we generalize the j-convex notion to a *multi-convex* shape and consider the existence of projections onto Π_n preserving this shape (such projections are said to be *multi-convex* projections). We then construct minimal norm multi-convex projections for (essentially) every multi-convex shape. Moreover, we do this for a large family of norms defined on $C^L[0,1]$ (integer L is greater than or equal $n-1$).

Following these introductory remarks, this paper is organized as follows. Section 2 defines multi-convexity (as a generalized j-convexity) and determines exactly when a multi-convex projection onto Π_n exists. Section 3 constructs minimal norm multi-convex projections for a family of norms defined on $C^L[0,1]$. In particular, we obtain the minimal norm results from [6] as a special case. Subsection 3.1 handles the case $n=2$ while 3.2 considers $n \geq 3$. The main result of Section 3 is Theorem 3.5; its proof is somewhat lengthy and as such has been organized into subsections 3.2.1 - 3.2.5. Section 4 provides technical details regarding existence of particular functionals in X^{**} as well as a represenation of elements from X^*.

2 K-convexity and multi-convexity

Let L and n denote positive integers such that $L \geq n-1$ (the reason for this inequality will be made clear). Let $C^L[0,1]$ denote the space of L-th continuously differentiable (real-valued) functions on $[0,1]$. Define on $C^L[0,1]$ the norms

$$\|f\|_L = \max_{i=0..L}\{\|f^{(i)}\|_\infty\}$$

and

$$\|f\|_{2,L} = \max\left\{\max_{j=0,3,4,\ldots,L-1}\{|f^{(j)}(0)|, |f^{(j)}(1)|\}, \max_{j=1,2,L}\{\|f^{(j)}\|_\infty\}\right\}.$$

$C^L[0,1]$ becomes a Banach space when endowed with either of these norms; we deote the respective spaces as X_L and $X_{2,L}$. Note that these norms are equivalent since

$$\left(\frac{2}{3}\right)^L \|\cdot\|_L \leq \|\cdot\|_{2,L} \leq \|\cdot\|_L. \tag{1}$$

NOTE 2.1 In general, given two norms that are equivalent (but not proportional), we should not expect a projection that has minimal operator norm with respect to first norm to be minimal in the operator norm determined by the second. However, this is exactly what we will discover to be true. In fact, we will prove minimality for an entire family of equivalent norms.

Let $\|\cdot\|$ denote any norm on $C^L[0,1]$ such that

$$\|f\|_{2,L} \leq \|f\| \leq \|f\|_L \tag{2}$$

for all $f \in C^L[0,1]$. Then $X = (C^L[0,1], \|\cdot\|)$ is a Banach space with dual space denoted by X^*. Note that, by the norm equivalence in (1), the dual space X^* remains unchanged as a set for any norm chosen according to (2). For integer $k \in [0, L]$ and $t \in [0,1]$ we denote by $\delta_t^k(f)$ the k-th derivative of f evaluated at t and regard $\delta_t^k \in X^*$. We will often express $\delta_t^k(f)$ using the familar notation $f^{(k)}(t)$.

Throughout this paper X will denote the Banach space $(C^L[0,1], \|\cdot\|)$ where $\|\cdot\|$ is as in (2). In the particular cases in which we endow $C^L[0,1]$ with $\|\cdot\|_{2,L}$ or $\|\cdot\|_L\|$, we denote the resulting Banach spaces as, respectively, $X_{2,L}$ and X_L.

In [6] it was proven that j-convexity can be preserved via a projection from X_L onto Π_n if and only if $j \geq n-1$. The notion of j-convexity is generalized in [10] in the following way. Let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ be an $(n+1)$-tuple with $\sigma_i \in \{0,1\}$; let $M = \max_{\sigma_i=1} i$. With $L \geq M$ define

$$S_{\boldsymbol{\sigma}} := \{f \in X \mid \sigma_i f^{(i)} \geq 0,\ i = 0, \ldots, n\}.$$

We say $f \in X$ is *multi-convex* if f belongs to the cone $S_{\boldsymbol{\sigma}}$. A projection from X onto $V = \Pi_n$ leaving invariant $S_{\boldsymbol{\sigma}}$ is called a *multi-convex projection.* We denote this set of projections by $\mathcal{P}_{S_{\boldsymbol{\sigma}}}$. Thus, for example, it follows from the results of [6] that $\mathcal{P}_{S_{\boldsymbol{\sigma}}} = \emptyset$ for every $\boldsymbol{\sigma} = \boldsymbol{e}_i$ where integer $i \in [0, n-2]$ and $\boldsymbol{e}_i = [0_0, 0_1, \ldots, 0_{i-1}, 1_i, 0_{i+1}, \ldots, 0_n]$.

For fixed $\boldsymbol{\sigma}$ let $M = \max_{\sigma_i=1} i$ and $m = \min_{\sigma_i=1} i$. We say that $\boldsymbol{\sigma}$ is *1-connected* if whenever $\sigma_i = \sigma_j = 1$ for $i < j$, we have $\sigma_k = 1$ for all $k = i, i+1, \ldots, j$. From here can characterize when a multi-convex shape can be preserved for any X. We begin with a result from [10] which describes the situation for the case of X_L.

THEOREM 2.1 (see [10]) *Let* $X_L = (C^L[0,1], \|\cdot\|_L)$. *Then* $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_L, V)$ $\neq \emptyset$ *if and only if* $M \geq n-1$ *and* $\boldsymbol{\sigma}$ *is 1-connected.*

This theorem is easily extended to the spaces we want to consider.

COROLLARY 2.1 *Let* $X = (C^L[0,1], \|\cdot\|)$ *for any norm* $\|\cdot\|$ *satisfying (2). Then* $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V) \neq \emptyset$ *if and only if* $M \geq n-1$ *and* $\boldsymbol{\sigma}$ *is 1-connected.*

Proof. From (2) if follows that, as sets, $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_L, V)$ and $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$ are identical. Thus from Theorem 2.1 the proof is complete. ■

3 Minimal multi-convex projections

In this section we obtain new (and summarize known) results for minimal multi-convex projections; i.e., minimal norm elements from $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$. As per Corollary 2.1, we assume $M \geq n-1$ and $\boldsymbol{\sigma}$ is 1-connected. Note by defintion, we always have $M \geq m$. Whenever $M = n$, we automatically assume $L \geq n$.

Our discussion can be organized using the ways in which m, M and n relate. Consider first the case where $m = M$. According to Corollary 2.1, there two possible situations of interest to us: they are $m = M = n-1$ and $m = M = n$. These cases are actually 'k-convex' shapes (regarded as specific multi-convex shapes); moreover these situations constitute somewhat extreme cases in the multi-convex realm. We consider in the sections below the case $m = M = n-1$ for all norms satisfying (2). In particular we obtain as a special case the results from [6] in the case of $\|\cdot\|_L$. In the case $m = M = n$, the minimal shape-preserving projection problem is completely unsolved for n$\geq$ 2 (the projection given in [6] partially solves the problem in the $n = 2$ case). Indeed, it is conjectured in [6] that a minimal norm projection from $X = (C^L[a,b], \|\cdot\|_L)$ onto $V = \Pi_n$ preserves n-convexity for every $L = 0, 1, \ldots$. That is, in the case of n-convexity, the minimal shape-preserving projection problem is perhaps equivalent to the minimal projection problem. As such, this paper does not address this case.

The case in which $m < n-1 \leq M$ is handled in [10]. We summarize the main results from that paper here as a single theorem.

THEOREM 3.1 (see [10]) *Let $n \geq 2$ and choose integer $k \geq 0$. Let $X = (C^{L+k}, \|\cdot\|)$, where $\|\cdot\|$ satisfies (2). Let $\boldsymbol{\sigma}$ be an $(n+k+1)$-tuple (with $\sigma_i \in \{0,1\}$) such that $\boldsymbol{\sigma}$ is 1-connected with $M \geq n+k-1$ and $m = k$. There there exists $P_0 \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_{n+k})$ such that $\|P_0\| = \sum_{j=0}^{n-1} \frac{1}{j!}$. Moreover, $\|P_0\| \leq \|P\|$ for all $P \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}$.*

The remaining case is $m = n-1$ and $M = n$ and this is considered below. It is interesting to note that method of proof required to establish minimal mulit-convex projections in this case differs substantially from the approaches in both [6] and [10]. Indeed, in [6] norm-minimality was obtained using the theory of best-approximation from a subspace, while in [10] minimality was (often) verified using a uniqueness arguement. Neither context is applicable for $m = n-1$ and $M = n$.

The arguments that follow rely on knowing the form of multi-convex projections. This form is described for the general shape-preserving setting in [13] and [4] and in the specific context of multi-convex preservation in Section 5 of [10]. We summarize the needed results in the following theorem. Note that for $u \in X^*$ and $v \in X$, we denote by $u \otimes v$ the operator mapping X into X by $(u \otimes v)(x) = u(x)v$.

THEOREM 3.2 (see [10]) *Let $n \geq 2$ and let $X = (C^L[0,1], \|\cdot\|)$ where $\|\cdot\|$ satisfies (2) and $L \geq n-1$. Let $\boldsymbol{\sigma}$ be 1-connected such that $M \geq n-1$ and $0 \leq m \leq M$ (note when $M = n$ we automatically assume $L \geq n$). Let $P \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$. Then P can be expressed as*

$$
\begin{aligned}
P &= u_0 \otimes 1 + u_1 \otimes \frac{x}{1!} + \cdots + u_{m-1} \otimes \frac{x^{m-1}}{(m-1)!} \\
&\quad + \delta_0^m \otimes \frac{x^m}{m!} + \delta_0^{m+1} \otimes \frac{x^{m+1}}{(m+1)!} + \cdots + \delta_0^{n-1} \otimes \frac{x^{n-1}}{(n-1)!} \\
&\quad + u_n \otimes \frac{x^n}{n!},
\end{aligned}
$$

for some $u_i \in X^*$, $i = 0, \ldots, m-1$ and, in case $M = n-1$,

$$u_n = \delta_1^{n-1} - \delta_0^{n-1};$$

otherwise $(M = n)$

$$u_n(f) = \int_{[0,1]} f^{(n)}(t)\ d\mu(t) \tag{3}$$

for some probablity Borel measure μ.

3.1 The quadratic case

In this section we consider the $n = 2$ case; i.e., projections onto $V = \Pi_2$ considered as a subspace of $X = (C^L, \|\cdot\|)$, where $\|\cdot\|$ satisfies (2). The results from here will be used in the following section to construct minimal norm multi-convex projections onto Π_n. We divide the results of the quadratic case into two theorems: one handling $n-1 = m = M-1$ and other $n-1 = m = M$.

THEOREM 3.3 *Let $\boldsymbol{\sigma}$ denote the 1-connected 3-tuple such that $m = 1$ and $M = 2$. Let $L \geq 2$ and $V = \Pi_2 \subset X = (C^L[0,1], \|\cdot\|)$. Define projection P_2 by*

$$P_2 = \frac{1}{2}(\delta_0 + \delta_1) \otimes 1 + \delta_0^1 \otimes (t - t^2/2 - 1/4) + \delta_1^1 \otimes (t^2/2 - 1/4). \quad (4)$$

Then $P_2 \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$ and $\|P_2\| = 3/2$. Moreover, for any $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$ $\|Q\| \geq \|P_2\|$.

Proof. Note that we may rewrite P_2 as

$$P_2 = \left(\frac{1}{2}(\delta_0 + \delta_1) - \frac{1}{4}(\delta_0^1 + \delta_1^1)\right) \otimes 1 + \delta_0^1 \otimes t + (\delta_1^1 - \delta_0^1) \otimes \frac{t^2}{2}; \quad (5)$$

from here it is easy to check that P_2 is a projection onto V. Applying Theorem 3.2 proves $P_2 \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$. Now consider any $f \in X$, $\|f\| = 1$. Then by (4)

$$\|P_2 f\| \leq 1 + \|t - t^2/2 - 1/4|_\infty + \|t^2/2 - 1/4\|_\infty = 3/2$$

and consequently, $\|P_2\| \leq 3/2$. To prove equality, we apply Theorem 4.2. Take any $F \in W_1$ (see Theorem 4.2 for the definition of W_1). Note that

$$P_2^{**}(F)1 = F(\delta_o^0 + \delta_1^0)/2 + F(\delta_o^1)/4 + F(\delta_1^1)/4 = 3/2,$$

as required.
To prove that P_2 is an operator of minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$ take $G \in Z_1$ corresponding to F via Theorem 4.2. Define

$$Z = \frac{(F, \delta_1^0) + (G, \delta_0^0)}{2}, \quad (6)$$

where

$$(F, \delta_1^0)(L) = (L^{**}F)(1)$$

and

$$(G, \delta_0^0)(L) = (L^{**}G)(0)$$

for any $L \in \mathcal{L}(X, V)$. It is clear that Z is a continuous, linear functional on $\mathcal{L}(X, V)$ of norm one. Moreover, since $F \in W_1$ and $G \in Z_1$,

$$Z(P_2) = \|P_2\| = 3/2.$$

To end the proof, we show that for any $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$,

$$Z(Q - P_2) \geq 0. \tag{7}$$

Fix $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, V)$. Using the form described in Theorem 3.2 (together with $m = 1$ and $M = 2 = n$) we may write

$$Q = u_0 \otimes 1 + \delta_0^1 \otimes t + u_2 \otimes t^2/2. \tag{8}$$

We will also express P_2 in this form: using (5) we write

$$P_2 = \phi_0 \otimes 1 + \delta_0^1 \otimes t + (\delta_1^1 - \delta_0^1) \otimes t^2/2. \tag{9}$$

By Lemma 4.1

$$u_0 = \sum_{i=0}^{L-1} \alpha_i \delta_0^i + \alpha^L,$$

and

$$\phi_0 = \sum_{i=0}^{L-1} \beta_i \delta_0^i + \beta^L.$$

Here α^L (β^L resp.) denotes the functional determined by a signed Borel measure α on [0,1] (signed Borel measure β on $[0, 1]$, resp.), i.e.,

$$\alpha^L(f) = \int_{[0,1]} f^{(L)}(t)\ d\alpha(t)$$

and

$$\beta^L(f) = \int_{[0,1]} f^{(L)}(t)\ d\beta(t)$$

for any $f \in X$. Since Q and P_2 are projections, we must have

$$\alpha_0 = \beta_0 = 1. \tag{10}$$

Now, using (8) and (9), observe that

$$Z(Q-P) = \frac{(F+G)(u_0 - \phi_0)}{2} + \frac{F(u_2 - (\delta_1^1 - \delta_0^1))}{4}.$$

Consider the first term of this sum, which, by (10), is equal to

$$(F+G)\left(\frac{\sum_{i=1}^{L-1} \alpha_i \delta_0^i + \alpha^L - \sum_{i=1}^{L-1} \beta_i \delta_0^i - \beta^L}{2}\right).$$

By Theorem 4.2 applied to F and G we see that this first term is equal to zero. Hence

$$Z(Q-P) = \frac{F(u_2 - (\delta_1^1 - \delta_0^1))}{4}.$$

But note the form of u_2 given 3; for every $f \in X$ we have

$$u_2(f) = \int_{[0,1]} f^{(2)}(t)\ d\mu(t).$$

If $L = 2$ then $F(u_2) \geq 0$ directly from Theorem 4.2; moreover, it is clear from the proof of Theorem 4.1 (in particular from (40)) that this inequality prevails for $L \geq 2$. Thus, using properties of F we find

$$Z(Q-P) = \frac{F(u_2 - (\delta_1^1 - \delta_0^1))}{4} = \frac{F(u_2)}{4} \geq 0$$

which completes the proof. ■

THEOREM 3.4 *Let $\boldsymbol{\sigma}$ denote the 1-connected 3-tuple such that $m = M = 1$. Let $L \geq 1$ and $V = \Pi_2 \subset X = (C^L[0,1], \|\cdot\|)$. Define projection P_2 by*

$$P_2 = \frac{1}{2}(\delta_0 + \delta_1) \otimes 1 + \delta_0^1 \otimes (t - t^2/2 - 1/4) + \delta_1^1 \otimes (t^2/2 - 1/4).$$

Then $P_2 \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X,V)$ and $\|P_2\| = 3/2$. Moreover, for any $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X,V)$, $\|Q\| \geq \|P_2\|$.

Proof. If $L \geq 2$ then this Theorem is an immediate of Theorem 3.3. So to finish the proof, we will assume that $L = 1$. Define for any $k \in \mathbb{N}$, $k \geq 3$, $f_{k,1}(0) = f_{k,1}(1) = 1$, $f_{k,1}(t) = 0$ for $t \in [1/k, 1 - 1/k]$ and in the linear way on the intervals, $[0, 1/k]$, $[1 - 1/k, 1]$. Set $g_{k,1} = -f_{k,1}$,

$$f_k(t) = 1 + \int_{[0,t]} f_{k,1}(s)ds,$$

and

$$g_k(t) = 1 + \int_{[0,t]} g_{k,1}(s)ds.$$

By definition of f_k and g_k, $\|g_k\| \to 1$ and $\|f_k\| \to 1$. By the Banach-Alaoglu theorem, the sequence $\{f_k\}$ has an accumulation point $F \in X^{**}$, $\|F\| \leq 1$, and the sequence $\{g_k\}$ has an accumulation point $G \in X^{**}$, $\|G\| \leq 1$. By the definition of f_k and g_k,

$$F(\delta_0^0) = F(\delta_1^0) = G(\delta_0^0) = G(\delta_1^0) = 1,$$

$$F(\delta_0^1) = F(\delta_1^1) = -G(\delta_0^1) = -G(\delta_1^1) = 1$$

and for any Borel measure on $[0, 1]$,

$$F(u^1) = -G(u^1).$$

Here $u^1 \in X^*$ is defined by

$$u^1(f) = \int_{[0,1]} f^{(1)}(t)du(t).$$

The rest of the proof is identical in form to the proof of Theorem 3.3 so we omit it. ■

3.2 The general case

In this section we construct a minimal norm multi-convex projection onto Π_n for the multi-convex shapes described by $m = n - 1 \leq M \leq n$ (i.e., the cases for which $\boldsymbol{\sigma}$ is 1-connected and either $m = M = n - 1$ or $m = n - 1$ and $M = n$).

NOTE 3.1 The following theorem generalizes Theorem 4.2 from [6] as well as corrects two arguments made in the proof of the Theorem. Specifically, the bound of $\frac{9}{8}$ given in inequality (26) below was incorrectly stated to be 1 in [6]. Additionally, the proof of Theorem 4.2 from [6] neglects to consider the $n = 3$ case. The proof below verifies this case in the generalized setting of this paper.

THEOREM 3.5 *Let $n \geq 3$ and $L \geq n-1$. Let $X = (C^L, \|\cdot\|)$ where $\|\cdot\|$ satisfies (2). Let $\boldsymbol{\sigma}$ denote a 1-connected (n+1)-tuple such that $m = n-1 \leq M \leq n$. Let P_2 denote the projection from Theorems 3.3 and 3.4. Define operator $P_n : X \to \Pi_n$ by*

$$P_n(f)(t) = tf(1)+(1-t)f(0)+\int_0^t P_{n-1}(f')(s)\,ds - t\int_0^1 P_{n-1}(f')(s)\,ds. \quad (11)$$

Then $P_n \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ and $\|P_n\| = \frac{3}{2}$. Moreover, for any $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$, $\|Q\| \geq \|P_n\|$.

Proof. We begin by using induction to show $P_n \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$. Suppose $P_{n-1} \in \mathcal{P}_{S_{\boldsymbol{\sigma}_1}}(X, \Pi_{n-1})$, where $\boldsymbol{\sigma}_1$ is the 1-connected n-tuple such that the position of the left-most 1 is n-2 ($= m-1$) and the right-most is M-1. Under this assumption it follows from the defintion of P_n that P_n is a projection onto Π_n. To verifiy that P_n preserves shape, let $f \in S_{\boldsymbol{\sigma}}$; i.e. $f^{(n-1)} \geq 0$ and (if $M = n$) $f^{(n)} \geq 0$. Note, by definition, this implies $f' \in S_{\boldsymbol{\sigma}_1}$. We claim $P_n f \in S_{\boldsymbol{\sigma}}$. For integer $k \geq 2$ and $g \in X$ observe that

$$(P_n g)^{(k)}(t) = (P_{n-1}g')^{(k-1)}(t). \quad (12)$$

Thus by our assumption on P_{n-1} we have

$$(P_n f)^{(n-1)}(t) = (P_{n-1}f')^{(n-2)}(t) \geq 0$$

and, if $M = n$,

$$(P_n f)^{(n)}(t) = (P_{n-1}f')^{(n-1)}(t) \geq 0$$

which verifies our claim. Let $n = 3$; from our previous section, we have $P_2 \in \mathcal{P}_{S_{\boldsymbol{\sigma}_1}}(X, \Pi_2)$ and therefore $P_n \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ for every integer $n \geq 3$.

The next step is to prove

$$\|P_n\| = 3/2 \quad (13)$$

for every n. We first prove (13) for $X = (C^L[0,1], \|\cdot\|_*)$, where

$$\|\cdot\|_* = \|\cdot\|_{2,L} \quad \text{or} \quad \|\cdot\|_* = \|\cdot\|_L.$$

To accomplish this, we first must consider the case $n = 3$. Let $f \in B(X)$ and recall that $\|P_3 f\|_*$ requires us to consider the quantities $P_3 f$, $(P_3 f)'$, $(P_3 f)^{(2)}$ and $(P_3 f)^{(3)}$. Expanding the expression given in (11) we find

$$P_3 f(t) = tf(1) + (1-t)f(0) + (-\frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{3})f''(0) + (\frac{t^3}{6} - \frac{t}{6})f''(1)$$

and thus for all $t \in [0,1]$

$$|P_3 f(t)| \le 1 + |-\frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{3}| + |\frac{t^3}{6} - \frac{t}{6}| < \frac{3}{2}.$$

Postponing $(P_3 f)'$ momentarily, it is clear that

$$|(P_3 f)^{(2)}(t)| < \frac{3}{2} \quad \text{and} \quad |(P_3 f)^{(3)}(t)| < \frac{3}{2}.$$

Now note that $(P_3 f)'(t)$ is quadratic and

$$|(P_3 f)'(t)| \le |f(1) - f(0)| + |-\frac{t^2}{2} + t - \frac{1}{3}| + |\frac{t^2}{2} - \frac{1}{6}| \le 3/2.$$

However, using (11), we can define an operator $\widehat{Q}$ on X by rewriting $(P_3 f)'(t)$ as follows:

$$\begin{aligned}(P_3 f)'(t) &= f(1) - f(0) + (P_2 f')(t) - \int_0^1 (P_2 f')(s)\, ds \\ &= (P_2 f')(t) + \int_0^1 f'(s)\, ds - \int_0^1 (P_2 f')(s)\, ds \\ &=: Q(f')(t)\end{aligned}$$

and now simply define $\widehat{Q}(f) := Q(f')$ (thus in this process we have also defined $Q : (C^{L-1}, \|\cdot\|_*) \to \Pi_2$). Clearly

$$\sup_{f \in B(C^L)} (P_3 f)'(1) = \sup_{f \in B(C^L)} Q(f')(1) = \sup_{f \in B(C^{L-1})} (Qf)(1)$$

and now, as done in the proof of Theorem 3.3, we apply Theorem 4.2 (as well make use of (42)) to select $F \in W_1$ and obtain

$$\begin{aligned}
\sup_{f\in B(C^L)} (P_3 f)'(1) &= \sup_{f\in B(C^{L-1})} (Qf)(1) \\
&\geq (Q^{**}F)(1) \\
&= (P_2^{**}F)(1) + \int_0^1 F(s)\ ds - \int_0^1 (P_2^{**}F)(s)\ ds \\
&= \frac{3}{2} + 1 - \int_0^1 \left(s + \frac{1}{2}\right)\ ds \\
&= \frac{3}{2}.
\end{aligned}$$

Therefore

$$\|P_3\| = \frac{3}{2} = \sup_{f\in B(C^L)} (P_3 f)'(1). \tag{14}$$

We now proceed to verify (13) for $n \geq 4$. Again we divide the work into three separate considerations: $P_n f, (P_n f)'$ and $(P_n f)^{(j)}$, $j \geq 2$.

3.2.1 $P_n f$ using $\|\cdot\|_*$

For $f \in B(X)$ and $t \in [0,1]$, we claim

$$|P_n f(t)| \leq \frac{3}{2}. \tag{15}$$

To this end, we first establish an alternate form of P_n; note that we may rewrite (11) as

$$P_n(f)(t) = tf(1) + (1-t)f(0) + T(P_{n-1}f')(t)$$

where

$$T(x)(t) = \left(\int_0^t - t\int_0^1\right) x(s)\ ds.$$

T is linear in x and, as is easy to check, vanishes on constant functions. Moreover, we claim for $x \in C[0,1]$,

$$\|Tx\|_\infty \leq \frac{1}{2}\|x\|_\infty. \tag{16}$$

Indeed consider

$$\begin{aligned}
|T(x)(t)| &= \left|(1-t)\int_0^t x(s)\ ds - t\left(\int_0^1 x(s)\ ds - \int_0^t x(s)\ ds\right)\right| \\
&= \left|(1-t)\int_0^t x(s)\ ds - t\int_t^1 x(s)\ ds\right| \\
&\leq (1-t)t\|x\|_\infty + t(1-t)\|x\|_\infty \\
&= 2(1-t)t\|x\|_\infty \\
&\leq \frac{1}{2}\|x\|_\infty.
\end{aligned}$$

Recalling the form of P_2 given in (4), we write

$$P_2(f)(t) = (f(0)+f(1))q_0(t) + f'(0)q_1(t) + f'(1)q_2(t)$$

where

$$q_0(t) := \frac{1}{2},\ \ q_1(t) := t - \frac{t^2}{2} - \frac{1}{4} \text{ and } q_2(t) := \frac{t^2}{2} - \frac{1}{4}.$$

Repeating the result from Theorem 3.3 we have

$$|P_2 f(t)| \leq 3/2.$$

From here we have

$$P_3(f)(t) = f(1)t + f(0)(1-t) + f^{(2)}(0)T(q_1)(t) + f^{(2)}(1)T(q_2)(t) \tag{17}$$

and we know $|P_3(f)(t)| \leq 3/2$. Let $id(t) := t$. Note that

$$T\left(f(1)id(t) + f(0)(1-id(t))\right) = (f(1)-f(0))T(id)(t)$$

and thus for $n \geq 4$, we find

$$P_n(f)(t) = f(1)t + f(0)(1-t) \tag{18}$$

$$+\sum_{i=1}^{n-3}(f^{(i)}(1) - f^{(i)}(0))T^i(id)(t) \tag{19}$$

$$+ f^{(n-1)}(0)T^{n-2}(q_1)(t) + f^{(n-1)}(1)T^{n-2}(q_2)(t) \tag{20}$$

where $T^j(x)$ denotes j iterations of T applied to x. We are now ready to establish (15); note that the right-hand side of (18) is bounded by 1 for norm $\|\cdot\|$ satsifying (2). Regarding (19), note that $T(id)(t) = t^2/2 - t/2$ and thus

$$\|T(id)\|_\infty \leq \frac{1}{8} = |T(id)(\frac{1}{2})|. \tag{21}$$

This bound together with (16) gives

$$\|\sum_{i=1}^{n-3}(f^{(i)}(1)-f^{(i)}(0))T^i(id)(t)\|_\infty \leq \sum_{i=1}^{n-3} 2\frac{1}{2^{i+2}} = \frac{1}{2} - \frac{1}{2^{n-2}}.$$

To bound (20), we note

$$\|q_1\|_\infty = |q_1(0)| = \frac{1}{4} = |q_2(0)| = \|q_2\|_\infty$$

and thus again employing (16) we find

$$\|f^{(n-1)}(0)T^{n-2}(q_1)(t) + f^{(n-1)}(1)T^{n-2}(q_2)(t)\|_\infty \leq \frac{1}{2^{n-1}}.$$

Combining the bounds on (18), (19) and (20) we find

$$|P_n f(t)| \leq \|P_n f\|_\infty \leq 1 + \frac{1}{2} - \frac{1}{2^{n-1}}$$

which proves (15).

3.2.2 $(P_n f)'$ using $\|\cdot\|_*$

For $f \in B(X)$ and $t \in [0,1]$ we claim

$$|(P_n f)'(t)| \leq \frac{3}{2}. \tag{22}$$

We need only consider $n \geq 4$. Let us first express $P_n f(t)$ in a way similar to that of the above section:

$$\begin{aligned} P_n(f)(t) &= f(1)t + f(0)(1-t) + (f'(1)-f'(0))T(id)(t) \\ &\quad + \sum_{i=2}^{n-3}(f^{(i)}(1)-f^{(i)}(0))T^i(id)(t) \\ &\quad + f^{(n-1)}(0)T^{n-2}(q_1)(t) + f^{(n-1)}(1)T^{n-2}(q_2)(t) \end{aligned}$$

and then differentiate:

$$(P_n f)'(t) = f(1) - f(0) + (f'(1)-f'(0))(t-1/2) \tag{23}$$

$$\quad + \sum_{i=2}^{n-3}(f^{(i)}(1)-f^{(i)}(0))T^i(id)'(t) \tag{24}$$

$$\quad + f^{(n-1)}(0)T^{n-2}(q_1)'(t) + f^{(n-1)}(1)T^{n-2}(q_2)'(t). \tag{25}$$

We now bound each of these labeled quantities.

Starting with the right-hand side of (23), we claim

$$|f(1) - f(0) + (f'(1) - f'(0))(t - 1/2)| \leq 1 + \frac{1}{8} \tag{26}$$

whenever $f \in B(X)$ and $t \in [0, 1]$. Replacing $f(t)$ with $f(1 - t)$ or $-f(t)$ if necessary, we verify this claim assuming $f'(1) \geq |f'(0)|$. Consider first the in which $f'(0) \geq 0$. Let $\alpha = f'(0)$ and define $g : [0, 1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} \alpha + t & \text{if } t \leq 1 - \alpha \\ 1 & \text{if } 1 - \alpha \leq t \leq 1 \end{cases}. \tag{27}$$

Note that for any $t \in [0, 1]$, $|f'(t)| \leq g(t)$. Indeed if $t \in [1 - \alpha, 1]$ then $|f'(t)| \leq 1 = g(t)$; for $t \in [0, 1 - \alpha]$ suppose $|f'(t)| > \alpha + t$. Then

$$|f'(t) - f'(0)| \geq |f'(t| - |f'(0)| > \alpha + t - \alpha = t$$

and consequently there exists $c \in [0, 1]$ such that

$$|f^{(2)}(c)| = \frac{|f'(t) - f'(0)|}{t} > 1 \tag{28}$$

which contradicts $f \in B(X)$. Thus

$$\begin{aligned} |f(1) - f(0) + (f'(1) - f'(0))(t - \frac{1}{2})| &\leq |f(1) - f(0)| + \frac{1}{2}|f'(1) - f'(0)| \\ &= |\int_0^1 f'(t)\ dt| + \frac{1}{2}|f'(1) - \alpha| \\ &\leq \int_0^1 g(t)\ dt + \frac{1 - \alpha}{2} \\ &= \alpha + \alpha(1 - \alpha) + \frac{(1 - \alpha)^2}{2} + \frac{1 - \alpha}{2} \\ &= 1 + \frac{\alpha - \alpha^2}{2} \\ &\leq 1 + \frac{1}{8} \end{aligned}$$

which establishes (26) in the case $f'(0) \geq 0$. Assume now that $f'(0) < 0$. If $\int_0^1 f'(s)\ ds \geq 0$ then set $F_1(t) = f'(t) - f'(0)$ and note that

$$\int_0^1 F_1(s)\ ds \geq \int_0^1 f'(s)\ ds. \tag{29}$$

From here, define $F(t) = \int_0^t F_1(s)\, ds$; from the definition of F and (29), it follows that

$$|f(1) - f(0)| + \frac{1}{2}|f'(1) - f'(0)| \leq |F(1) - F(0)| + \frac{1}{2}|F'(1) - F'(0)|.$$

At this point, we set $\alpha = F'(0) = 0$ and proceed to define $g(t)$ as in (27). We then obtain

$$|F(1) - F(0)| + \frac{1}{2}|F'(1) - F'(0)| \leq 1 + \frac{1}{8}$$

as done above (note that we need $|F^{(2)}(t)| = |f^{(2)}(t)| \leq 1$ to obtain the contradiction in (28); but we see that this follows from the definition of $\|f\| \leq 1$). Finally, if $\int_0^1 f'(s)\, ds < 0$ then set $F_1(t) = -f'(t) + f'(1)$ and define $F(t) = \int_0^t F_1(s)\, ds$ and proceed as above to establish (26).

Regarding (24), first note that this expression has meaning only for $n \geq 5$ (the $n = 4$ case is covered below in our consideration of (25)). Let D denote differentiation with respect to t. Then we have

$$\left| \sum_{i=2}^{n-3} (f^{(i)}(1) - f^{(i)}(0)) D(T^i(id)(t)) \right| \leq \sum_{i=2}^{n-3} 2\|D(T^i(id)(t))\|_\infty.$$

A direct calculation verifies that

$$|D(T^2(id)(t))| \leq \frac{1}{12}, \ |D(T^3(id)(t))| \leq \frac{\sqrt{3}}{216} \ \text{ and } \ |D(T^4(id)(t))| \leq \frac{1}{720}.$$

We now work bound $|D(T^i(id)(t))|$ for $i \geq 5$. Applying T four times to id we find

$$|T^4(id)(t)| = \left| \frac{1}{120}t^5 - \frac{1}{48}t^4 + \frac{1}{72}t^3 - \frac{1}{720}t \right| \leq \frac{1}{2^{12}}.$$

Thus using $\|T^4(id)(t)\|_\infty < \frac{1}{2^{12}}$ and (16) we have, for $i \geq 4$,

$$\|T^i(id)(t)\|_\infty < \frac{1}{2^{i+8}}. \tag{30}$$

With $f \in X$ note that

$$\|D(T(f)(t))\|_\infty = \|T^{i-1}(f)(t) - \int_0^1 T^{i-1}(f)(s)\, ds\|_\infty$$

and therefore

$$\|D(T(f)(t))\|_\infty \leq 2\|f\|_\infty. \tag{31}$$

Using (31) in combination with (30) where $i \geq 5$ and $f = T^{i-1}(id)$ we find

$$\|D(T^i(id)(t))\|_\infty = \|DT(f)(t))\|_\infty \leq \frac{1}{2^{i+6}}.$$

Therefore

$$\sum_{i=5}^{n-3} 2\|D(T^i(id)(t))\|_\infty \leq \sum_{i=5}^{\infty} 2\frac{1}{2^{i+6}} = \sum_{i=10}^{\infty} \frac{1}{2^i} = \frac{1}{512}.$$

And finally

$$\sum_{i=2}^{n-3} 2\|D(T^i(id)(t))\|_\infty \leq \frac{1}{6} + \frac{\sqrt{3}}{108} + \frac{1}{360} + \frac{1}{512} < \frac{5}{24}$$

which provides a bound of $\frac{5}{24}$ for (24).

To bound (25), consider first $n = 4$ and let D denote differentiation with respect to t. A straightforward calculation shows that $\|D(T^2(q_i)\|_\infty \leq \frac{1}{24}$ for $i = 1, 2$ which implies a bound of $\frac{1}{12}$ on (25) for $n = 4$. Assume now that $n \geq 5$. We claim

$$|D(T^{n-2}(q_1)(t))| + |D(T^{n-2}(q_2)(t))| \leq \frac{1}{6}. \tag{32}$$

An straightforward calculation shows

$$\|T^2(q_i)\|_\infty \leq \frac{1}{24} \text{ for } i = 1, 2$$

and therefore for each integer $k \geq 2$ we have

$$\|T^k(q_i)\|_\infty \leq \frac{1}{2^{k-2}24} \text{ for } i = 1, 2.$$

From here we obtain (32) by applying (31) for each $i = 1, 2$:

$$\|D(T^{n-2}(q_i)(t))\|_\infty = \|D(T(T^{n-3}(q_i))(t))\|_\infty \leq 2\frac{1}{2^{n-5}24} \leq \frac{1}{12}$$

Combining the bounds on (23), (24) and (25) we find (for $n \geq 4$)

$$|(P_n f)'(t)| \leq 1 + \frac{1}{8} + \frac{5}{24} + \frac{1}{6} = \frac{3}{2}.$$

3.2.3 $(P_n f)^{(j)}$ using $\|\cdot\|_*$

For $j \geq 2$ and $t \in [0,1]$ we claim

$$\sup_{f\in B(X)} |(P_n f)^{(j)}(t)| \leq \frac{3}{2} \quad \text{for } j \neq n-2$$

and

$$\sup_{f\in B(X)} |(P_n f)^{(n-2)}(t)| \leq \sup_{f\in B(X)} |(P_n f)^{(n-2)}(1)| = \frac{3}{2}.$$

For $n = 4$ the claim is verified using (12) together with the result in (14). Assume the claim holds for $n-1$. Let X denote $(C^L[0,1], \|\cdot\|_*)$ and X_1 denote $(C^{L-1}[0,1], \|\cdot\|_{**})$ where $\|\cdot\|_{**}$ is either $\|\cdot\|_{L-1}$ or $\|\cdot\|_{2,L-1}$. Now for $j \neq n-2$ consider

$$\begin{aligned}
\sup_{f\in B(X)} |(P_n f)^{(j)}(t)| &= \sup_{f\in B(X)} |(P_{n-1} f')^{(j-1)}(t)| \\
&= \sup_{f\in B(X_1)} (P_{n-1} f)^{(j-1)}(t)| \\
&\leq \frac{3}{2}
\end{aligned}$$

by assumption. Similarly, for $j = n-2$ we find

$$\begin{aligned}
\sup_{f\in B(X)} |(P_n f)^{(n-2)}(t)| &= \sup_{f\in B(X_1)} (P_{n-1} f)^{(n-3)}(t)| \\
&\leq \sup_{f\in B(X_1)} (P_{n-1} f)^{(n-3)}(1)| \\
&= \frac{3}{2}
\end{aligned}$$

by assumption. This verfies our claim for all $n \geq 4$.

3.2.4 $\|P_n f\| \leq \frac{3}{2}$

The combination of Sections 3.2.1, 3.2.2 and 3.2.3 proves (13) for cases in which

$$X = (C^L[0,1], \|\cdot\|_*).$$

Let $\|\cdot\|$ denote any norm satisfying (2); let $X = (C^L[0,1], \|\cdot\|)$, $X_{2,L} = (C^L[0,1], \|\cdot\|_{2,L})$ and $X_L = (C^L[0,1], \|\cdot\|_L)$. Note by (2) and the norm definitions we have bounds

$$\sup_{f\in B(X)} \|P_n f\| \geq \sup_{f\in B(X_L)} \|P_n f\| \geq \sup_{f\in B(X_L)} \|P_n f\|_{2,L}$$

and

$$\sup_{f\in B(X)} \|P_n f\| \leq \sup_{f\in B(X_{2,L})} \|P_n f\| \leq \sup_{f\in B(X_{2,L})} \|P_n f\|_L.$$

Note that

$$\sup_{f\in B(X_L)} \|P_n f\|_{2,L} \geq \sup_{f\in B(X_L)} |(P_n f)^{(n-2)}(1)| = \frac{3}{2} \tag{33}$$

by the claim of Section 3.2.3. Furthermore, using the form of $P_n f$ given in (18), (19) and (20) as well the bounds established in Sections 3.2.1, 3.2.2 and 3.2.3, we have

$$\sup_{f\in B(X_{2,L})} \|(Pf)^{(j)}\|_\infty = \sup_{f\in B(X_L)} \|(Pf)^{(j)}\|_\infty \leq \frac{3}{2}$$

for every integer $j \in [0, L]$. Thus

$$\sup_{f\in B(X_{2,L})} \|P_n f\|_L \leq \frac{3}{2}. \tag{34}$$

Applying bounds (33) and (34) to $\|P_n f\|$ we find

$$\frac{3}{2} \leq \sup_{f\in B(X_L)} \|P_n f\|_{2,L} \leq \sup_{f\in B(X)} \|P_n f\| \leq \sup_{f\in B(X_{2,L})} \|P_n f\|_L \leq \frac{3}{2}$$

which establishes (13) for allowable $\|\cdot\|$.

3.2.5 minimality

Let $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ where $X = (C^L[0,1], \|\cdot\|)$. We claim

$$\|Q\| \geq \frac{3}{2}; \tag{35}$$

this will establish the minimality of $\|P_n\|$. Let $f \in B(X)$. From Theorem 3.2 we have

$$(Qf)^{(n-2)}(t) = u_{n-2}(f)1 + f^{(n-1)}(0)t + u_n(f)\frac{t^2}{2}$$

where

$$u_n(f) = \int_{[0,1]} f^{(n)}(t)\ d\mu(t)$$

and μ is Lebesgue measure in the case $M = n-1$ and a Borel measure otherwise. Since Q is a projection, $u_{n-2} \perp \Pi_k$ for integer $k = 0, \ldots, n-3, n-1, n$ and $u_{n-2}(t^{n-2}) = 1$. Thus since

$$f(t) = \int_0^t \int_0^{t_{n-3}} \ldots \int_0^{t_1} f^{(n-2)}(s)\ ds\ dt_1 \ldots dt_{n-3} + p(t)$$

where $p(t) \in \Pi_{n-3}$ we may write

$$u_{n-2}(f) = u_{n-2}\left(\int_0^t \int_0^{t_{n-3}} \ldots \int_0^{t_1} f^{(n-2)}(s)\ ds\ dt_1 \ldots dt_{n-3}\right).$$

Define

$$\widehat{u}_{n-2}(g) = u_{n-2}\left(\int_0^t \int_0^{t_{n-3}} \ldots \int_0^{t_1} g(s)\ ds\ dt_1 \ldots dt_{n-3}\right).$$

Similarly, define

$$\widehat{u}_{n-1}(g) = g'(0)$$

and

$$\widehat{u}_n(g) = \int_{[0,1]} g^{(2)}(t)\ d\mu(t).$$

Let $\widehat{Q} := \widehat{u}_{n-2} \otimes 1 + \widehat{u}_{n-1} \otimes t + \widehat{u}_n \otimes \frac{t^2}{2}$ and note

$$(Qf)^{(n-2)}(t) = \widehat{Q}(f^{(n-2)})(t). \tag{36}$$

Moreover, $\widehat{Q}$ is a projection from $C^2[0,1]$ onto Π_2 which preserves either 1-convexity ($M = 1$) or the monotone-convex shape ($m = 1$ and $M = 2$) (indeed this follows immediately from either (36) or the form of $\widehat{Q}$ defined on $C^2[0,1]$). Thus, for integer $K \geq 2$, if we regard $\widehat{Q} : X_{2,K} \to \Pi_2$ then

$$\begin{aligned}\|\widehat{Q}\| &= \sup_{f \in B(X_{2,K})} \|\widehat{Q}f\|_{2,K} \\ &= \sup_{f \in B(X_{2,K})} \max\{|\widehat{Q}(f)(0)|, |\widehat{Q}(f)(1)|, \|(\widehat{Q}f)'\|_\infty, \|(\widehat{Q}f)''\|_\infty\} \\ &\geq \frac{3}{2}.\end{aligned}$$

From the form of $\widehat{Q}$ it is clear that

$$\|\widehat{Q}\| = \sup_{f\in B(X_{2,K})} \max\{|\widehat{Q}(f)(0)|, |\widehat{Q}(f)(1)|\}.$$

In fact, from (6) and (7) it follows that we replace $B(X_{2,K})$ with simply $B(X_K)$ to obtain

$$\sup_{f\in B(X_K)} \max\{|\widehat{Q}(f)(0)|, |\widehat{Q}(f)(1)|\} \geq \frac{3}{2}.$$

And finally we can estimate the norm of $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$:

$$\begin{aligned}
\sup_{f\in B(X)} \|Qf\| &\geq \sup_{f\in B(X_L)} \|Qf\|_{2,L} \\
&\geq \sup_{f\in B(X_L)} \max\{|(Qf)^{(n-2)}(0)|, |(Qf)^{(n-2)}(1)|\} \\
&= \sup_{f\in B(X_L)} \max\{|\widehat{Q}(f^{(n-2)})(0)|, |\widehat{Q}(f^{(n-2)})(1)|\} \\
&= \sup_{f\in B(X_{L-n+2})} \max\{|\widehat{Q}(f)(0)|, |\widehat{Q}(f)(1)|\} \\
&\geq \frac{3}{2}.
\end{aligned}$$

This establishes (35); thus P_n given in (11) has minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$.

4 Technical results: norms and representations in X^*

THEOREM 4.1 *Let* $X = (C^{L+1}[0,1], \|\cdot\|)$, $L \geq 1$, $g = \sum_{i=0}^{L+1} \delta_0^i$, *and* $h = \delta_0^0 - \sum_{i=1}^{L+1} \delta_0^i$. *Define*

$$W = \{F \in X^{**} : F(g) = L+2, \|F\| = 1\}$$

and

$$Z = \{F \in X^{**} : F(h) = L+2, \|F\| = 1\}.$$

Assume u *is a Borel measure on [0,1]. Define* $u^L \in X^*$ *by*

$$u^L(f) = \int_{[0,1]} f^{(L)}(t)du(t).$$

Then for any $F \in W$ and for any Borel measure u on [0,1],

$$F(u^L) \geq 0. \tag{37}$$

Moreover, for any $F \in W$ there exists $G \in Z$ such that

$$G(u^L) = -F(u^L) \tag{38}$$

for any Borel measure u on [0,1].

Proof. By Corollary 4.1 from [10] $W \neq \emptyset$. Fix $F \in W$ and a Borel measure u. By the Goldstine Theorem there exists a sequence $\{f_l\} \subset X$, $\|f_l\| \leq 1$, such that

$$f_l(\delta_o^{(i)}) = f_l^{(i)}(0) \to F(\delta_o^{(i)}) = 1 \tag{39}$$

for $i = 0, ..., L+1$ and

$$f_l(u^{(L)}) = \int_{[0,1]} f_l^{(L)}(t) du(t) \to F(u^{(L)}). \tag{40}$$

In particular, $\|f_l^{(L+1)}\|_\infty \leq 1$. Hence by the Mean Value Theorem for any $s, t \in [0,1]$ $l \in \mathbb{N}$,

$$|f_l^{(L)}(s) - f_l^{(L)}(t)| \leq |t - s|.$$

Also $\|f^{(L)}\|_\infty \leq 1$. By the Ascoli-Arzela Theorem, passing to a subsequence, if necessary, we can assume that there exists $f \in C[0,1]$ such that

$$\|f_l^{(L)} - f\|_\infty \to 0. \tag{41}$$

Now we show that $f(t) \geq 0$ for any $t \in [0,1]$. By (39), $f(0) = 1$. Assume on the contrary, that there exists $t_o \in (0,1]$ such that $f(t_o) < 0$. By (39) there exists $\delta > 0$ such that

$$|f_l^{(L)}(0) - f_l^{(L)}(t_o)| > 1 + \delta.$$

for $l \geq l_o$. By The Mean Value Theorem,

$$1 + \delta < |f_l^{(L)}(0) - f_l^{(L)}(t_o)| \leq \|f_l^{(L+1)}\|_\infty t_o \leq 1;$$

a contradiction. Hence $f(t) \geq 0$ for any $t \geq 0$. By (41)

$$f_l(u^{(L)}) = \int_{[0,1]} f_l^{(L)}(t) du(t) \to \int_{[0,1]} f(t) du(t) \geq 0,$$

since f is nonnegative and u is a measure. By (40),

$$F(u^{(L)}) = \int_{[0,1]} f(t)du(t) \geq 0,$$

which completes the proof of the first part of our theorem.

Now for fixed $F \in W$, we construct $G \in Z$ satisfying (38) for any Borel measure u on [0,1]. First we show that for any $t \in [0,1]$

$$F(\delta_t^{(i)}) = 1 \tag{42}$$

for any $i = 0, ..., L-1$. Without loss of generality, we can assume that $i = L-1$. Since $F \in W$, $F(\delta_o^{(L-1)}) = 1$. Assume, on the contrary that $F(\delta_{t_o}^{(L-1)}) < 1-\delta$ for some $t_o \in (0,1]$. By the Mean Value Theorem, for $l \geq l_o$

$$f_l^{(L-1)}(t_o) - f_l^{(L-1)}(0) = f_l^{(L)}(s_l)(t_o - 0).$$

Hence $f_l^{(L)}(s_l) < -\delta/2$ for $l \geq l_o$. Since by assumptions, $L \geq 1$, applying once more the Mean Value Theorem, we get for $l \geq l_o$

$$1 + \delta/2 < |f_l^{(L)}(s_l) - f_l^{(L)}(0)| = |f_l^{(L+1)}(w_l)|s_l.$$

But this implies that $\|f_l^{(L+1)}\|_\infty > 1 + \delta/2$ for $l \geq l_o$; a contradiction. Thus (42) follows.

Now fix a net $\{f_\beta\} \subset X$, $\|f_\beta\| \leq 1$ such that

$$f_\beta \to F$$

weakly* in X^{**}. (By the Goldstine Theorem such a net exists.) Let us define for any β and $t \in [0,1]$,

$$g_{\beta,L+1}(t) = -f_\beta^{(L+1)}(t), \tag{43}$$

$$g_{\beta,j}(t) = -f_\beta^{(j)}(0) + \int_{[0,t]} g_{\beta,j+1}(s)ds \tag{44}$$

for $j = L, ..., 1$ and

$$g_\beta(t) = f_\beta(0) + \int_{[0,t]} g_{\beta,1}(s)ds \tag{45}$$

Now we show that $\|g_\beta\| \to 1$. First we prove that $\|g_\beta\|_\infty \to 1$. Note that for any $t \in [0,1]$

$$f_\beta(t) = f_\beta(0) + \int_{[0,t]} f_\beta^{(1)}(s)ds. \tag{46}$$

By (43) and the previous part of the proof for any $t \in [0,1]$

$$f_\beta(t) \to F(\delta_t^{(o)}) = 1$$

for any $t \in [0,1]$. By (46)

$$\int_{[0,t]} f_\beta^{(1)}(s)ds \to 0.$$

Since for $i = 1, ...L+1$ $g_{\beta,i} = -f_\beta^{(i)}$,

$$\int_{[0,t]} g_\beta^{(1)}(s)ds \to 0,$$

for any $t \in [0,1]$. Consequently, by (45),

$$\|g_\beta\|_\infty \to 1,$$

as required. Note that for any $i = 1, ..., L+1$,

$$g_\beta^{(i)} = g_{\beta,i} = -f_\beta^{(i)}.$$

Hence $\|g_\beta\| \to 1$, as required. By the Banach-Alaoglu Theorem, $\{g_\beta\}$ has a cluster point $G \in X^{**}$, $\|G\| \leq 1$. By the construction of functions g_β, $G \in Z$. Moreover, for any Borel measure u on $[0,1]$

$$G(u^L) = \lim_\beta g_{\beta,L}(u^L) = -\lim_\beta f_\beta^{(L)}(u^L) = -F(u^L),$$

which completes the proof. ■

THEOREM 4.2 *Let* $X = (C^L[0,1], \|\cdot\|)$, $L \geq 1$, $g = \sum_{i=0}^L \delta_0^i$ *and* $h = \delta_0^0 - \sum_{i=1}^L \delta_0^i$. *Set*

$$W_1 = \{F \in X^{**} : F(g) = L+1, \|F\| = 1\}$$

and

$$Z_1 = \{G \in X^{**} : G(h) = L+1, \|G\| = 1\}.$$

Then $W_1 \neq \emptyset$ and $Z_1 \neq \emptyset$. Moreover, there exist $F \in W_1$ and $G \in Z_1$ such that for any Borel measure u on [0,1]

$$F(u^L) \geq 0, \tag{47}$$

$$G(u^L) = -F(u^L), \tag{48}$$

$$F(\delta_1^0) = G(\delta_1^0) = 1, \tag{49}$$

and

$$F(\delta_1^i) = -G(\delta_1^i) = 1 \tag{50}$$

for $i = 1, ..., L-1$, if $L \geq 2$. Here $u^L \in X^*$ is defined by

$$u^L(f) = \int_{[0,1]} f^{(L)}(t)du(t).$$

Also

$$F(m_t^L) = 0 \text{ for any } t \in [0,1], \tag{51}$$

where

$$m_t^L(f) = \int_{[0,t]} f^{(L)}(t)dm(t)$$

and m is the Lebesgue measure on $[0,1]$.

Proof. Note the differences between this theorem and Theorem 4.1: in addition to (51), see that (47) and (48) use the highest derivative possible in X, where Theorem 4.1 uses the next-to-highest derivative possible.

Let $X_1 = (C^{L+1}[0,1], \|\cdot\|)$. By Theorem 4.1 applied to X_1, there exist $F_1 \in W$ and $G_1 \in Z$ satisfying (37) and (38) for any Borel measure u on [0,1]. By the Goldstine Theorem applied to $B(X_1^{**})$, there exists a net $\{f_\beta\} \subset X_1$, $\|f_\beta\| \leq 1$ for any β such that $f_\beta \to F_1$ weak-$*$ in X_1^{**}. Analogously, there exists a net $\{g_\beta\} \subset X_1$, $\|g_\beta\| \leq 1$ tending to G_1 weak-$*$ in X_1^{**}. Since $X_1 \subset X$, as sets, $\{f_\beta\} \subset X$ and $\{g_\beta\} \subset X$. Moreover each function f_β and g_β has norm at most one in X, since its norm in X_1 is at most one. By the Banach-Alaoglu Theorem applied to $B(X^{**})$ the set $\{f_\beta\}$ has an accumulation point $F \in B(X^{**})$ and the set $\{g_\beta\}$ has an accumulation point $G \in B(X^{**})$. Since $F_1 \in W$, and $G_1 \in Z$, we have $F \in W_1$ and $G \in Z_1$ (and thus neither W_1 nor Z_1 is empty, in particular in the case $L = 1$).

Moreover, by Theorem 4.1 applied to F_1 and G_1, for any Borel measure u on $[0,1]$

$$0 \le F_1(u^L) = \lim_\beta (\int_{[0,1]} f_\beta^{(L)}(t) du(t)) = F(u^L)$$

which verifies (47) and

$$\begin{aligned} F(u^L) &= \lim_\beta (\int_{[0,1]} f_\beta^{(L)}(t) du(t)) \\ &= F_1(u^L) = -G_1(u^L) = -\lim_\beta (\int_{[0,1]} g_\beta^{(L)}(t) du(t)) \\ &= -G(u^L), \end{aligned}$$

which verifies (48). With regard to (49), we get $F(\delta_1^0) = 1$ immediately from (42). From (45) we find $g_\beta(t) = 2f_\beta(0) - f_\beta(t)$ and therefore

$$G(\delta_1^0) = \lim_\beta g_\beta(1) = \lim_\beta (2f_\beta(0) - f_\beta(1)) = 1,$$

proving (49). When $L \ge 2$ we can apply (42) to F_1 and easily obtain (50).

To establish (51), choose any $t \in [0,1]$ and note $F(m_t^L) \ge 0$ by (47). By the Fundamental Theorem of Calculus, for any $f \in X$,

$$m_t^L(f) = \int_{[0,t]} f^{(L)}(t) dm(t) = f^{(L-1)}(t) - f^{(L-1)}(0).$$

Since $F \in W_1$,

$$0 \le F(m_t^L) = F(\delta_t^{(l-1)} - \delta_0^{(L-1)}) = F(\delta_t^{(L-1)}) - 1.$$

Hence $F(\delta_t^{(L-1)}) \ge 1$. Since $\|F\| = 1$,

$$1 = F(\delta_t^{(L-1)})$$

and thus (51) follows. ■

LEMMA 4.1 *Let $u \in X^*$ where $X = (C^L[0,1], \|\cdot\|)$ and $\|\cdot\|$ satifies (2). There there exists constants c_i, $i = 0, \dots L-1$ and a signed (Borel) measure μ on $[0,1]$ such that*

$$u(f) = \sum_{i=0}^{L-1} c_i f^{(i)}(0) + \int_0^1 f^{(L)}(s)\ d\mu(s)$$

for every $f \in X$.

Proof. Let

$$c_i := \frac{u(x^i)}{i!} \text{ for } i = 0, \ldots, L-1.$$

Define the functional $\widehat{u}$ on $(C[0,1], \|\cdot\|_\infty)$ by

$$\widehat{u}(g) = u\left(\int_0^x \int_0^{x_1} \cdots \int_0^{x_{L-1}} g(s)\ ds\ dx_{L-1} \ldots dx_1\right).$$

Note that $\widehat{u} \in (C[0,1], \|_\infty)^*$ and thus there exists a signed Borel measure μ on $[0,1]$ such that

$$\widehat{u}(g) = \int_0^1 g(s)\ d\mu(s) \tag{52}$$

for all $g \in (C[0,1], \|_\infty)$. Let $f \in X$; thus $f^{(L)} \in (C[0,1], \|_\infty)$. Therefore

$$\begin{aligned}\widehat{u}(f^{(L)}) &= u\left(\int_0^x \int_0^{x_1} \cdots \int_0^{x_{L-1}} f^{(L)}(s)\ ds\ dx_{L-1} \ldots dx_1\right) \\ &= u(f(x)) - f(0)c_0 - f^{(1)}(0)c_1 - f^{(2)}(0)c_2 - \ldots - f^{(L-1)}c_{L-1}.\end{aligned}$$

Solving this last equation for $u(f(x))$ and applying (52) we find

$$u(f(x)) = \sum_{i=0}^{L-1} c_i f^{(i)}(0) + \int_0^1 f^{(L)}(s)\ d\mu(s).$$

■

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